Symbolic dynamics II
The stadium billiard

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ABSTRACT

We construct a well ordered symbolic dynamics plane for the stadium billiard. In this symbolic plane the forbidden and the allowed orbits are separated by a monotone pruning front, and allowed orbits can be systematically generated by sequences of approximate finite grammars.

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1 Introduction

The stadium billiard was introduced by Bunimovich[1] as an example of a system with non-vanishing Kolmogorov entropy. The stadium billiard has also been used as an example of a chaotic quantum system[2]. The wave functions in the quantized system show complicated patterns[3] and the stadium billiard is a model for which scars are conjectured to exist[4]. The distribution of quantum eigenvalues is in good agreement with a GOE distribution[5].

The stadium is a convex shaped two dimensional area, limited by two half-circles with radius 1 joined by two straight parallel lines of length $2a$, fig. 1. A point particle moves with constant velocity in this area and is reflected elastically when it reaches the border of the area. The straight lines can be removed and replaced by an infinite chain of semicircles as shown in fig. 2. The two chains of semicircles are separated by the distance $2a$ and the same orbits exist in this billiard as in the stadium.

A symbolic dynamics description of the stadium billiard has not yet been showed to give an efficient way of computing thermodynamic measures. For other systems a proper understanding of symbolic dynamics has been very useful in semiclassical calculations[6] and we expect the same to apply to the stadium billiard. There have been several attempts to develop a good symbolic description of all classical orbits in the stadium[7, 8] but the symbolic description turns out to be more complicated than for other systems, such as the logistic map[9], the Hénon map[10, 11] and the disk billiards[12, 13].

A symbolic description of a dynamical system assigns to each orbit in the system a sequence of symbols. The symbols are elements of a finite or infinite alphabet. A covering alphabet assigns distinct symbol sequences to distinct orbits. Given a covering alphabet, it still may be true that a given symbol sequence corresponds to no dynamical orbit in the system; such a sequence is called a pruned symbol sequence[11]. We shall here distinguish between pruning that depends on the value of a parameter and pruning that is parameter independent. The parameter in the stadium is a, the half length of the straight edge. While in the stadium we have both kinds of pruning, there also exist systems with only the parameter dependent pruning. This is the case for the Hénon map. For sufficiently large parameter a, the Hénon map is a complete Smale horseshoe[14] and can be described by a binary alphabet without any pruning. Also the 3-disk billiard with the distance between the disks sufficiently large can be described by a complete binary alphabet[13].

When the parameter a in the stadium billiard goes to infinity we have the stadium with only parameter independent pruning, and we shall show here which orbits are forbidden in this limit. With decreasing parameter a, further orbits are

pruned. To describe this pruning we construct a new *well ordered* alphabet and determine a *pruning front*. This pruning front is a monotone curve in a symbol plane and the front changes monotonously with the parameter value.

2 Covering alphabet for stadium billiard

The symbolic alphabet is $s_t \in \{1, 2, 3, 4, 5, 6\}$ where each symbol corresponds to one bounce of the particle off a wall in the stadium. The symbols are indicated in fig.1 and are similar to the symbols defined in ref.[7]:

1. A bounce in the lower horizontal line.
2. A bounce in the upper horizontal line.
3. A bounce in the right semicircle, anticlockwise with respect to the center of this semicircle.
4. A bounce in the left semicircle, anticlockwise with respect to the center of this semicircle.
5. A bounce in the right semicircle, clockwise with respect to the center of this semicircle.
6. A bounce in the left semicircle, clockwise with respect to the center of this semicircle.

We us the conjecture in ref.[7] that no two distinct orbits can be described by the same symbol sequence in this alphabet which has been tested numerically by Biham and Kvale.

An orbit in the stadium corresponds to an infinite symbol sequence. The particle bouncing at time $t = 0$ with the symbol s_0, has the symbolic description $\cdots s_{-2}s_{-1}s_0s_1s_2\cdots$. Let x be the distance measured along the border of the stadium anticlockwise from the center point on the lower straight line with $0 \leq x \leq 4a + 2\pi$. Let ϕ be the angle between the normal vector to the wall and the outgoing velocity measured anticlockwise $-\pi/2 \leq \phi \leq \pi/2$. The position x and the angle ϕ of the bounce at time $t = 0$ can be determined if the orbit exists and we know the infinite symbolic description of the orbit. The only exception is the periodic orbit $\overline{12} = \ldots 121212 \ldots$ which has $\phi = 0$ and any position on the straight lines.

A special case are the orbits going through the center of a semicircle. These orbits have neither a clockwise nor an anticlockwise bounce, so there are two symbols that may represents the same orbit. The symbolic past of these orbits are identical with the time reversed symbolic future, and orbits with this symmetry in the symbols are counted twice.

3 Parameter independent pruning

The parameter independent pruning rules are determined by simple geometrical considerations. Biham and Kvale have described some of these rules which they call geometrical pruning rules in ref.[7]. There cannot be two successive bounces off the same straight line, so the symbol strings 11 and 22 are forbidden. A clockwise bounce cannot be followed by an anticlockwise bounce in the same semicircle and a clockwise bounce cannot be followed by an anticlockwise bounce. This forbids the symbol strings 35, 53, 46 and 64. These are the only forbidden strings of length two, see table 1.a.

An orbit that bounces first off the right semicircle and then off a straight line cannot return to the right semicircle without bouncing in the left semicircle first. This gives rise to a second kind of pruning rules whith strings like 315 and 313 forbidden. All these forbidden right–middle–right and left–middle–left symbol strings are listed in table 1.b.

The third kind of parameter independent pruned orbits can be described by using the picture of the chain of semicircles in fig. 2. The figure shows an orbit that first bounces clockwise on the right side and then bounces clockwise off a semicircle situated two semicircles below the first one and on the left side of the billiard. The same orbit is drawn in the stadium in fig. 1 and has the symbolic description $S = 5126$. In fig. 2 it is clear that the outgoing line from the second bounce has to be above the incoming line since the bounce is clockwise. In the symbolic description of the stadium, the string $S = 5126$ cannot be followed by any of the symbol strings: _23, _25, _5, _3, _13, _15, _125, _123, _1213, _1215, _12125, ... More complicated examples of orbits pruned this way can also be constructed. This third kind of pruning rules are complicated to state in symbols s_t while in the well ordered symbols introduced below, the rule is quite simple.

This third kind of pruning rules also prevents a periodic orbit which bounces normal to the semicircle to have more than two symbolic representations. These periodic orbits have an infinite number of bounces normal to the semicircle but one is however not allowed to choose randomly the symbol for each normal bounce. When the symbol for one normal bounce is choosen there is only one single choice for future and past symbols as all other choices are pruned by the third kind of parameter independent pruning rules.

4 Well ordered alphabet for stadium billiard

The rule that the symbol strings 11, 22, 35, 46, 53 and 64 are forbidden, makes it possible to construct a new alphabet with 5 symbols. Each new symbol is constructed from a combination of two bounces. As for any symbol s one of the possible next symbols is forbidden, only 5 different symbols can follow the symbol s.

We demand that the new alphabet is well ordered[12, 13]; i.e., that there exists a natural ordering of the symbols which preserves the topological ordering of orbits in the phase space (x, ϕ). Table 2 gives new symbols $v_t \in \{0, 1, 2, 3, 4\}$, constructed from all the possible combinations of symbol pairs $s_{t-1}s_t$. Table 2 is obtained by observing the change of symbols s_t when ϕ increases. Let the bounce be on the lower straight line $s_t = 1$. In the limit $\phi = -\pi/2$ the next bounce is a anticlockwise bounce in the right semicircle giving $s_{t+1} = 3$ and when ϕ increases this bounce becomes normal to the wall and then clockwise giving $s_{t+1} = 5$. Increasing ϕ further gives $s_{t+1} = 2$, then $s_{t+1} = 4$ and then in the end $s_{t+1} = 6$ in the limit $\phi = \pi/2$. This gives the ordered two symbol combinations $\{13, 15, 12, 14, 16\}$ which is the first column in table 2. The same method gives the ordered combinations for the other bounces. By fixing the angle ϕ and changing the position x we find that the ordering of two symbol combinations is the same but we do not necessarily find all the 5 combinations for one value of ϕ. This is a consequence of the pruning.

As shown in ref.[13], the ordering is conserved for a bounce off a focusing (convex) wall while the ordering is reversed for a bounce off a straight wall. The reverse ordered symbol is $v'_t = 4 - v_t \in \{0, 1, 2, 3, 4\}$. The new well-ordered symbol w_t is equal to v_t or v'_t, depending on whether the number of preceeding bounces off the straight edges is odd or even. If p_t is the number of symbols 1 and 2 in the sequence $s_0 s_1 s_2 \ldots s_{t-1}$, the well ordered symbols are given by

$$w_t = \begin{cases} v_t & \text{if } p_t \text{ odd} \\ 4 - v_t & \text{if } p_t \text{ even} \end{cases} \tag{1}$$

We now use the symbols w_t in order to define a real number in base 5

$$\gamma = 0.w_1 w_2 w_3 \ldots = \sum_{t=1}^{\infty} \frac{w_t}{5^t} \tag{2}$$

The real number γ, where $0 \leq \gamma \leq 1$, is the symbolic coordinate representation of the future of the orbit starting at the point (x, ϕ).

We also need the symbolic past of the orbit. When we follow an orbit backward in time, the clockwise and anticlockwise bounces exchange roles. As our convention we choose the time reversed sequence $\hat{s}_0 \hat{s}_1 \hat{s}_2 \ldots \hat{s}_t \ldots$ which is related to the forward symbol sequence $\ldots s_{-t} \ldots s_{-2} s_{-1} s_0$ as follows:

s_{-t}	1	2	3	4	5	6
$\hat{s}_t$	1	2	5	6	3	4

With this convention we find that the values $\hat{w}_t$ are constructed from $\hat{s}_t$ as in table 2 and we get the real number

$$\delta = 0.\hat{w}_0 \hat{w}_1 \hat{w}_2 \ldots = \sum_{t=0}^{\infty} \frac{\hat{w}_t}{5^{t+1}} \tag{3}$$

The coordinate δ is the symbolic representation of the past of the orbit where $0 \leq \delta \leq 1$. The square (δ, γ) is the symbolic plane representation of the possible

stadium billiard trajectories. The value of γ is increasing with increasing value of x and with increasing value of ϕ. The value of δ is increasing with increasing value of x, but decreasing with increasing value of ϕ.

The phase space for the straight edge and the phase space for the semicircle are different so we have to work in two different symbol planes for the two cases. The symbol plane for the straight line is denoted (γ_l, δ_l) and the symbol plane for the semicircle is denoted (γ_c, δ_c).

5 Parameter independent pruning in the symbol plane

The parameter independent pruned orbits discussed above are easily described in the symbol planes. The right–middle–right and left–middle–left forbidden orbits in table 1.b are represented in the symbol plane (γ_l, δ_l) by all the points in the two squares $1/2 < \gamma_l < 1$ and $1/2 < \delta_l < 1$, and $0 < \gamma_l < 1/2$ and $0 < \delta_l < 1/2$. In fig. 3 a) we plot the points (γ_l, δ_l) corresponding to all bounces off a straight line of a particle bouncing 10^5 times in the billiard with $a = 5$. As expected, no points are inside the two forbidden squares.

The third kind of parameter independent pruned orbits is easily described in the semicircle symbol plane. In the symbol plane for $s_0 = 5$ or $s_0 = 6$, these forbidden orbits are represented as the triangle $\delta_c \geq \gamma_c + 1/5$. The rule follows from comparing a string in the future $s_0 s_1 s_2 \ldots$ with a string in the past $\hat{s}_0 \hat{s}_1 \hat{s}_2 \ldots$. If the bounce is 5 or 6 then the symbolic future coordinate should be smaller than the symbolic past cordinate to reflect that the outgoing angle is positive while the incoming angle is negative. The limit case is when the angle goes to 0 (a bounce normal to the wall) where the future symbols $s_0 s_1 s_2 s_3 \ldots$ is identical to the past symbols $\hat{s}_0 s_1 s_2 \ldots$, with exception for the first symbol which is time reversed. From table 2 we find that this difference in the first symbol gives a difference in symbolic value $\delta_c - \gamma_c = 1/5$. Since the symbolic values are well ordered, the forbidden strings satisfy

$$\delta_c \geq \gamma_c + 1/5 \tag{4}$$

Fig. 3 b) shows the bounces of the long chaotic orbit in the symbol plane for $s_t = 5$, and as expected, there are no points above the line $\delta_c = \gamma_c + 1/5$. If $s_t = 3$ or $s_t = 4$, the symbol plane is the same as for $s_t = 5$ or $s_t = 6$ but reflected with respect to the line $\gamma = \delta$; that is the same as reversing the time. In this space the forbidden triangle is given by

$$\delta_c \leq \gamma_c - 1/5 \tag{5}$$

The parameter independent pruning can now be summerized as follows: forbidden symbol sequences of length 2 (table 1.a) are removed by introducing the new

alphabet; forbidden sequences of the second kind (table 1.b) are squares in the symbol plane (γ_l, δ_l); forbidden sequences of the third kind are triangles in the symbol plane (γ_c, δ_c). Unfortunately the description of the second and the third kind of pruning becomes complicated if we try to describe the second kind in (γ_c, δ_c), or the third kind in (γ_l, δ_l).

6 Parameter dependent pruning

Some orbits cannot exist in the stadium if the distance between the two semicircles is sufficiently short. These orbits are called dynamically pruned orbits in ref.[7]. As the parameter a decreases, infinite families of orbits are pruned. We explain the mechanism of this pruning by an example.

If $a > 1$ the two periodic orbits $\overline{S_1} = \overline{335553}$ and $\overline{S_2} = \overline{321521}$ exist and are drawn in fig. 4 a). If $a = 1$, both orbits bounce in the singular point joining the straight edge and the semicircle, and are not distinguishable in the phase space, see fig. 4 b), even though the two orbits have different stabilities and different symbolic description. For $a < 1$, these two periodic orbits do not exist. As shown by Biham and Kvale[7], these forbidden orbits can be traced by using whole circles instead of the semicircles, and extending the straight edges beyond the tangencies with the circles. Figs.4 c) and 4 d) show the two (unphysical) periodic orbits when $a < 1$.

This example shows how orbits are pruned when a point in the orbit reaches the singular point at the junction of the semicircle and the straight edge. The pruning front is determined by finding the symbol sequences of all orbits bouncing off this singular point, with different angles ϕ. In the symbol planes these orbits give monotone curves as shown in figs. 5 a) and 5 b). The monotonicity follows directly from the definition of the well ordered symbols. The two pruning fronts in the two symbol spaces are obtained by choosing either the symbol for the straight edge or the symbol for the semicircle as the symbol s_0 describing the bounce off the singular point. The pruning front is not a continuous curve but a Cantor set of points, with the gaps in the pruning front generated by the images of the forbidden regions in the symbol plane. Any continuous monotone curve going through all points in the pruning front may be used as a continuous pruning front.

All bounces of an admissible orbit in the stadium give points on one side of the pruning front. We refer to the region on the other side of the pruning front as the *primary* pruned region. Figs. 6 a) and 6 b) show bounces from a long chaotic orbit and a comparison with the pruning fronts in fig. 5 shows, as expected, that the orbit never enters the primary pruned region.

The parameter dependent pruned areas in the symbol plane are: the primary pruned region, its images by symmetry, and the images of these regions by iteration forward and backward in time. Iteration in time is a shift operation but complicated by the fact that one iterates either to the same symbol plane, or to the other symbol

plane, depending on the point. A shift operations is always more complicated in well ordered symbols than in the original symbols[16]. One of the symbol plane symmetries is the symmetry between (γ, δ) and $(1-\gamma, 1-\delta)$ which follows from the symmetry between odd and even in the algorithm (1). In the (γ_l, δ_l) symbol plane, we also have a symmetry between (γ_l, δ_l) and (δ_l, γ_l) which arises from the time reversal symmetry between an orbit and the time reversed orbit. This symmetry relates the point (γ_c, δ_c) for bounce with symbol 5 or 6, and the point (δ_c, γ_c) for a bounce with symbol 3 or 4. The symbol describing the bounce off a semicircle changes with time reversal since a symbol depends on clockwise or anticlockwise motion.

The primary pruned region can be converted into forbidden symbol sequences in different approximate ways. We have choosen to find the completely forbidden sequences of finite length by the method described in ref.[13]. Table 3 gives all completely forbidden substrings S of length less or equal to than 5 for parameter $a = 1$. Longer forbidden substrings can be found from fig. 5.

Orbits are pruned only when a decreases, the primary pruned region grows monotonously with decreasing a, and consequently the topological entropy is decreasing with a.

An orbit is never pruned alone. An orbit starting from the singular point can always be represented either with the symbol for a straight line bounce or a semicircle bounce. A periodic orbit bifurcates together with a infinite family of orbits with similar symbolic description. The example discussed above gives the family

$$\ldots A_1 B_1 5 C_1 D_1 4 A_2 B_2 5 C_2 D_2 4 \ldots$$

with $A_i \in \{1,4\}$, $B_i \in \{2,5\}$, $C_i \in \{1,5\}$ and $D_i \in \{2,4\}$. This family of orbits exists only when $a \geq 1$. At $a = 1$ all the points (γ, δ) for each orbit in this family are on the pruning front and this gives a constraint on the pruning front. Each orbit bifurcates in a family like this at a singular parameter value. In ref.[15] we show that for dispersive billiards the singular bifurcation in a billiard can be related to a bifurcation tree in a smooth potential and we expect this to be true also for a focusing billiard, such as the stadium.

7 Conclusion

We have shown that the symbolic alphabet introduced in ref.[7] enable us to decide if an orbit is admissible or not. This is accomplished by constructing new symbols that define a well ordered symbol plane. Meiss[8] have showed that those orbits never bouncing off the straight lines are ordered in space by a rotation number, while we have showed that all unstable orbits in the stadium are ordered in space by using the right symbolic dynamics. In the symbol plane there are a few connected

regions that are forbidden, and an orbit is admissible if and only if no bounce of the orbit has a point in the forbidden regions. We expect that for no parameter values there is a complete alphabet or a finite Markov partition, but sequences of approximate Markov partition can be constructed using the methods presented here.

1. The stadium billiard and the 6 symbols s_t. Symbols 3 and 4 correspond to anticlockwise while 5 and 6 correspond to clockwise bounces. The thin lines trace an orbit with symbolic dynamics $\ldots 5126 \ldots$.

2. The unfolded stadium with two infinite chains of semicircles. The thin lines are the same orbit as in fig. 1.

3. All the bounces of a chaotic orbit bouncing 10^5 times in the stadium with parameter $a = 5$ plotted in the symbol plane. a) The symbol plane for the straight edges. b) The symbol plane for the clockwise semicircular edges.

4. The two orbits $\overline{335553}$ (dotted line) and $\overline{321521}$ (solid line) in the stadium. a) The two legal orbits, $a = 2$. b) The two orbits at the pruning point, $a = 1$. c) The two forbidden orbits for $a = 0.6$. d) Blow-up of the upper right corner of c) showing the forbidden bounce off the straight edge.

5. The pruning front in the symbol plane for parameter $a = 1$. a) The symbol plane for the straight edge. b) The symbol plane for the clockwise semicircular edge.

6. All bounces of a chaotic orbit bouncing $30,000$ times in the stadium with parameter $a = 1$, plotted in the symbol plane. a) The symbol plane for straight edges. b) The symbol plane for the clockwise semicircular edge.

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$s_t s_{t+1}$
11
22
35
53
46
64

a)

s
$32^i(12)^k1^j3$
$32^i(12)^k1^j5$
$52^i(12)^k1^j3$
$52^i(12)^k1^j5$
$42^i(12)^k1^j4$
$42^i(12)^k1^j6$
$62^i(12)^k1^j4$
$62^i(12)^k1^j6$

b)

Table 1: Two kind of symbol sequences that are always forbidden in the stadium billiard, $i, j \in \{0, 1\}$ and $k \in \{0, 1, 2, \ldots\}$.

$s_{t-1}s_t$	v_t
13	0
15	1
12	2
14	3
16	4

$s_{t-1}s_t$	v_t
24	0
26	1
21	2
23	3
25	4

$s_{t-1}s_t$	v_t
33	0
32	1
34	2
36	3
31	4

$s_{t-1}s_t$	v_t
44	0
41	1
43	2
45	3
42	4

$s_{t-1}s_t$	v_t
52	0
54	1
56	2
51	3
55	4

$s_{t-1}s_t$	v_t
61	0
63	1
65	2
62	3
66	4

Table 2: Construction of the well ordered alphabet in the stadium billiard. The well ordered symbols w_t are constructed by choosing $w_t = v_t$ when the number of 1's and 2's (bouncing in a straight line) in the preceeding symbol string (including s_{t-1}) is *odd* and choosing $w_t = 4 - v_t$ when the number is *even*.

$w_{-1}w_0w_1w_2$
$44 \cdot 22$
$44 \cdot 21$
$44 \cdot 20$
$34 \cdot 22$
$34 \cdot 21$
$24 \cdot 44$
$14 \cdot 44$
$04 \cdot 44$
$24 \cdot 43$
$14 \cdot 43$

a)

$s_{-2}s_{-1}s_0s_1s_2$				
44121	44123	44125	24121	24123
44421	44423	44425	24421	24423
66212	66215	66213	16212	16215
66612	66615	66613	16612	16615
55121	55126	55124	25121	25126
55521	55526	55524	25521	25526
33212	33214	33216	13212	13214
33312	33314	33316	13312	13314

b)

Table 3: The parameter dependent completly pruned subsequences of length ≤ 5 for parameter $a = 1$. a) Symbols w_t. b) Symbols s_t.